

On the motion of classical three-body system with consideration of quantum fluctuations

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Abstract

We obtained the system of stochastic differential equations which describes the classical motion of the three-body system under influence of quantum fluctuations. Using SDEs, for the joint probability distribution of the total momentum of bodies system were obtained the partial differential equation of the second order. It is shown, that the equation for the probability distribution is solved jointly by classical equations, which in turn are responsible for the topological peculiarities of tubes of quantum currents, transitions between asymptotic channels and, respectively for arising of quantum chaos.

I. INTRODUCTION

The behavior of quantum systems, non-integrable in the classical limit was discussed still at the dawn of development of the quantum mechanics¹. The occurrence of wave mechanics has allowed us to formulate the time evolution of any quantum system in the framework of the time-dependent Schrödinger equation. Despite the fact that in quantum mechanics essentially possible only a statistical description due to the linearity of the Schrödinger equation, the wave function of a quantum system has a deterministic behavior from the time. Last circumstance creates serious difficulties for the description of the dynamical system at transition to the classical limit where the system demonstrates a chaotic behavior. In classical mechanics, the main cause arising of chaotic motion is the exponentially rapid divergence of nearby trajectories that in quantum mechanics becomes impossible by reason of the Heisenberg uncertainty relation². It is obvious, that in a standard representation of the quantum mechanics, by analogy with the classical mechanics it is impossible to make enter the criteria of the exponentially rapid divergence of close wave functions since as already mentioned they are deterministic functions. Moreover as it can be shown the analogue of Arnold's theorem on quantum mapping, different from the zero Planck's constant promotes to the suppression of chaos⁴. In case when n -dimensional classical dynamical system in the phase space exhibits chaotic behavior and the sizes of the chaotic regions in the phase space is lesser than \hbar^n , then its quantum analogue "does not see" such regions and, respectively commits regular movement. In any case, the problem arises in the limit of the classical motion when $\hbar \rightarrow 0$, and respectively any classical system in the result of such transition, by reason of Arnold's theorem necessarily becomes integrable, that generally speaking is incorrect. Finally, the description of a quantum system is seriously complicated, when sizes of regions of the classical chaotic motion become larger, than the volume of the quantum cell \hbar^n . In this case obviously chaos should be appear also in the motion of the quantum system, i.e. should be chaotic the wave function.

In recent years many studies on the problem of the quantum chaos in systems with autonomous and nonautonomous Hamiltonians have been conducted (see for example^{2,3,5-9}), however, the above problems as well as a number other issues regarding to the foundations of quantum mechanics, still remain unresolved.

As well-known the general three-body problem is a typical example of a dynamic system

where on a large regions of the phase space are observed all features of a complex motion including the bifurcation and chaos. In this paper, we will consider the problem of multichannel scattering in the classical three-body system taking into account the random external factors, in particular quantum fluctuations, without using perturbation methods. It should be noted that such consideration of the problem is very interesting, from point of view its wide applications in applied problems, for example at simulation of elementary chemical reactions taking into account randomness of an environment etc. Also it is important for a deeper understanding the foundations of quantum mechanics, namely the principle of Bohrs correspondence between the quantum system and, with its non-integrable classical analog.

Note that the general three-body classical problem concerns the question of understanding motions of three arbitrary point masses traveling in space according to Newton's laws of mechanics. Many works on celestial and analytical mechanics, stellar and molecular dynamics, devoted to the study of this problem are, as a rule, carried out by numerical simulation (see for example¹⁰⁻¹⁴).

Recently the author proved¹⁵, that the general classical three-body problem may be reduced to the system of sixth order on the hypersurface of the energy, which has the conform-Euclidean metric. The first three equations of this system form the closed system of nonlinear partial differential equations of the first order (the system of the Riccati's type equations). The main idea of the work is that the equations become random due to random exposure of the environment. Mathematically, this is equivalent to assuming that the metric of space is random, and respectively a motion of three-body system is described by the system of the stochastic differential equations (SDEs) of Langevin type. ‘

Finally, using the system SDEs, we derive the evolution equation describing the quantum currents of the scattering process in the momentum representation, which is solved in combination with the system of classical equations of three-body. In the work the criteria for the occurrence of quantum chaos is formulated.

II. THE CLASSICAL THREE-BODY SYSTEM

The classic three-body problem in a most general formulation, is the problem of multichannel scattering, with series of possible asymptotic outcomes. Schematically, the scatter-

ing process can be represented as:

$$1 + (23) \longrightarrow \left\{ \begin{array}{l} 1 + (23), \\ 1 + 2 + 3, \\ (12) + 3, \\ (13) + 2, \\ (123)^* \longrightarrow \left\{ \begin{array}{l} 1 + (23), \\ 1 + 2 + 3, \\ (12) + 3, \\ (13) + 2, \\ (123)^{**} \rightarrow \left\{ \dots \right. \end{array} \right. \end{array} \right.$$

where 1,2 and 3 are the separate particles, the bracket (.) denotes bound states, while "*" and "**" denote some transition states of three-body system.

The aim of this study is the obtaining the equation describing the quantum probabilistic currents going between asymptotic states.

The classical Hamiltonian of three-body system after Jacobi and mass-scale¹⁶ transformations can be written as (see also¹⁷):

$$H(\mathbf{r}; \mathbf{p}) = \frac{\mathbf{p}^2}{2\mu_0} + V(\mathbf{r}), \quad (1)$$

where $\mathbf{r} = \mathbf{r} \oplus \mathbf{R} \in \mathbb{R}^6$ and $\mathbf{p} \in \mathbb{R}^6$ are correspondingly the position vector and the momentum of the effective mass (imaginary point):

$$\mu_0 = [m_1 m_2 m_3 / (m_1 + m_2 + m_3)]^{1/2}.$$

Recall that m_1, m_2 and m_3 are masses of corresponding bodies). With respect of the coordinate \mathbf{r} , that it represents the distance between 2 and 3 particles, while the coordinate \mathbf{R} designates the distance between the particle 1 and the center of mass of the pair (2,3) (see Fig. 1), in addition the total potential $V(\mathbf{r})$ depends from distances between particles, that means that the interaction potential in fact depends from three variables.

Let us consider the following system of coordinates:

$$\rho_1 = r = \|\mathbf{r}\|, \quad \rho_2 = R = \|\mathbf{R}\|, \quad \rho_3 = \theta, \quad \rho_4 = \Theta, \quad \rho_5 = \Phi, \quad \rho_6 = \Psi, \quad (2)$$

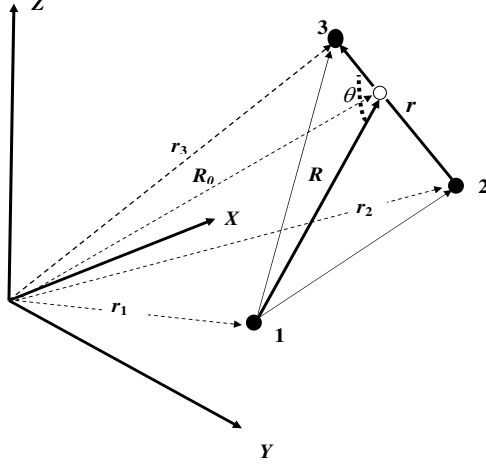


FIG. 1. The Cartesian coordinate system where the set of vectors $\mathbf{r}_1, \mathbf{r}_2$ and \mathbf{r}_3 denotes coordinates of the 1, 2 and 3 particles, respectively. The circle "o" denotes the center-of-mass of pair (12) which in the Cartesian system is expressed by \mathbf{R}_0 . The Jacobi coordinates system described by the radius-vectors \mathbf{R} and \mathbf{r} , in addition to θ , denote scattering angle.

where the first set of three coordinates; $\{\bar{\rho}\} = (\rho_1, \rho_2, \rho_3)$ describes a position of an imaginary point on the plane formed by bodies triangle (internal coordinates), while; $\Theta \in (-\pi, +\pi]$, $\Phi \in (-\pi, +\pi]$ and $\Psi \in [0, \pi]$ are Euler angles describing rotation of the plane in 3D space (external coordinates). It is clear that the full potential $V(\mathbf{r})$ in the new coordinates will depend only on internal coordinates $\{\bar{\rho}\}$.

The kinetic energy of three-body system in these variables has the form (see also¹⁸):

$$T = \frac{1}{2\mu_0} \{ \dot{\mathbf{r}}^2 + \dot{\mathbf{R}}^2 \} = \frac{1}{2\mu_0} \left\{ \dot{r}^2 + r^2 [\times \mathbf{k}]^2 + (\dot{\mathbf{R}} + [\omega \times \mathbf{R}])^2 \right\}, \quad (3)$$

where $\{\varrho\}$ the direction of unit vector \mathbf{k} in the moving reference frame is defined by expression; $\mathbf{R}\mathbf{R}^{-1} = \pm \mathbf{k}$. Below the vector $\mathbf{k} = (0, 0, 1)$ is directed towards a positive direction of the axis $0Z$, while the angular velocity ω describes rotation of the frame $\{\varrho\}$ with respect to laboratory system. Having done a simple calculations in the expression (3) we can find:

$$T = \frac{1}{2\mu_0} \left\{ \dot{r}^2 + \dot{R}^2 + R^2 \dot{\theta}^2 + Ar^2 + BR^2 \right\}, \quad (4)$$

where the following designations are made:

$$A = (\dot{\Theta}^2 + \dot{\Phi}^2 \sin^2 \Theta) = \omega_X^2 + \omega_Y^2, \quad B = (\omega_X \cos \theta - \omega_Z \sin \theta)^2.$$

Let us note that at deriving the expression (4) we have used definition of the moving system $\{\varrho\}$ by the requirement that unit vector γ lies in the plane OXZ at the angle θ to OZ , i.e. $\gamma = (\sin \theta, 0, \cos \theta)$. As regards of projections of an angular velocity they satisfy equations:

$$\begin{aligned}\omega_X &= \dot{\Phi} \sin \Theta \sin \Psi + \dot{\Theta} \cos \Psi, \\ \omega_Y &= \dot{\Phi} \sin \Theta \cos \Psi - \dot{\Theta} \sin \Psi, \\ \omega_Z &= \dot{\Phi} \cos \Theta - \dot{\Psi}.\end{aligned}\tag{5}$$

Taking into account (4) and (5) we can find the metric tensor:

$$\gamma^{\alpha\beta} = \begin{pmatrix} \gamma^{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma^{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma^{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma^{44} & \gamma^{45} & \gamma^{46} \\ 0 & 0 & 0 & \gamma^{54} & \gamma^{55} & \gamma^{56} \\ 0 & 0 & 0 & \gamma^{64} & \gamma^{65} & \gamma^{66} \end{pmatrix}\tag{6}$$

where the following designations are made:

$$\begin{aligned}\gamma^{11} &= \gamma^{22} = 1, \quad \gamma^{33} = R^2, \quad \gamma^{44} = r^2 + R^2 \cos^2 \Psi \cos^2 \theta, \\ \gamma^{55} &= r^2 \sin \Theta + R^2 (\sin^2 \Theta \sin^2 \Psi \cos^2 \theta + \cos^2 \Theta \sin^2 \theta - 2^{-1} \sin 2\Theta \sin 2\theta \sin \Psi) \\ \gamma^{66} &= R^2 \sin^2 \theta, \quad \gamma^{45} = \gamma^{54} = R^2 (\sin \Theta \sin 2\Psi \cos^2 \theta - 2 \cos \Theta \cos \Psi \sin 2\theta), \\ \gamma^{46} &= \gamma^{64} = R^2 \sin 2\theta \cos \Psi, \quad \gamma^{56} = \gamma^{65} = R^2 (\sin \Theta \sin \Psi \sin 2\theta - 2 \cos \Theta \sin^2 \theta).\end{aligned}$$

Without going into details let us note that the considered problem has 12 integrals of motion using which the initial 18th order system is reduced to the 8th order system¹⁹.

III. CLASSICAL THREE-BODY PROBLEM AS THE PROBLEM OF GEODESIC FLOWS ON ENERGY HYPERSURFACE

As it is easy to see, the classical system of three bodies at motion in the 3D Euclidean space permanently forms the triangle, and hence Newton's equations describe the dynamical system on the space of such triangles¹⁸. The last means that we can formally consider the

motion of a body-system consisting of two parts. The first is the rotational motion of the body-triangle in the 3D Euclidian space and the second is the internal motion of bodies on the plane defined by the triangle. Mathematically, the configuration manifold of solid body R^6 can be represented as a direct product of two subspaces²⁰:

$$\mathbb{R}^6 :\Leftrightarrow \mathbb{R}^3 \times S^3,$$

where \mathbb{R}^3 is the manifold which is defined as an orthonormal space of relative distances between bodies while S^3 denotes the space of the rotation group $SO(3)$. However in the considered problem, connections between bodies are not holonomic and respectively we must change the representation for the configuration manifold $\mathbf{M} :\Leftrightarrow \mathbb{R}^6$.

Let us consider the region of localization of dynamical system (further named *the internal space* \mathcal{M}_t):

$$x^1 = ||\mathbf{r}||, \quad x^2 = ||\mathbf{R}||, \quad x^3 = ||\mathbf{r} + \mathbf{R}|| = \sqrt{(x^1)^2 - 2x^1x^2 \cos \theta + (x^2)^2}, \quad (7)$$

where $\theta \in [0, \pi]$ is the angle between the vectors $\mathbf{r} \in [0, \infty)$ and $\mathbf{R} \in [0, \infty)$ is the scattering angle in Jacobe coordinate system. The set of *internal coordinates* $\{\bar{x}\} = (x^1, x^2, x^3) \in \mathcal{M}_t$. The rotation of a plane defined by body-triangle will be described by the set of three *external coordinates* $(x^4, x^5, x^6) \in S_t^3$, where S_t^3 is a space of the rotation group $SO(3)$ in a neighborhood of interior points $M_i\{(x^1, x^2, x^3)_i\} \in \mathcal{M}_t$.

The subset of all interior points $\check{\mathbf{M}} \subset \mathbf{M}$ is represented as:

$$\check{\mathbf{M}} \cong \mathcal{M}_t \times S_t^3.$$

The set $\mathbf{M} \setminus \check{\mathbf{M}}$ has zero measure however in some cases it can be important for dynamics of the classical three-body system.

So, we can define a local system of coordinates in which further studies will be carried:

$$\overline{x^1, x^6} = \{x\} \in \check{\mathbf{M}}. \quad (8)$$

Taking into account the Krylov's well-known work²¹, we will study the motion of three-body system on the hypersurface of potential energy (HPE) of bodies system. Note that the HPE is the curved space the metric tensor of which is defined as follows:

$$g_{\mu\nu}(\{x\}) = g(\{x\})\delta_{\mu\nu}, \quad g^{\mu\nu} = \delta^{\mu\nu}/g(\{x\}), \quad g(\{x\}) = [E - U(\{x\})]U_0^{-1} > 0, \quad (9)$$

where E and $U(\{x\}) \equiv V(\mathbf{r})$ are the total energy and total interaction potential of bodies system respectively, $\delta_{\mu\nu}$ is the Kronecker symbol and $U_0 = \max ||U(\{x\})||$ denotes maximal depth of the potential. In the case the total potential depends on the relative distances between the particles; $V(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = V(|\mathbf{r}_1 - \mathbf{r}_2|, |\mathbf{r}_1 - \mathbf{r}_3|, |\mathbf{r}_2 - \mathbf{r}_3|)$, then the metric tensor can be written in the internal space as; $g_{\mu\nu}(\{x\}) = g_{\mu\nu}(\{\bar{x}\})$.

Now using the variational principle of Maupertuis we can derive geodesic equations^{20,22}:

$$\ddot{x}^\alpha + \Gamma_{\beta\gamma}^\alpha \dot{x}^\beta \dot{x}^\gamma = 0, \quad \alpha, \beta, \gamma = \overline{1, 6}, \quad (10)$$

where $\dot{x}^\alpha = dx^\alpha/ds$ and $\ddot{x}^\alpha = d^2x^\alpha/ds^2$; in addition s is a scalar parameter of motion (e.g. the proper time), the Christoffel symbol; $\Gamma_{\beta\gamma}^\alpha(\{x\}) = \frac{1}{2}g^{\alpha\mu}(\partial_\gamma g_{\mu\beta} + \partial_\beta g_{\gamma\mu} - \partial_\mu g_{\beta\gamma})$, where $\partial_\alpha \equiv \partial_{x^\alpha}$.

Taking into account the definition for the metric tensor (9) from (10) we can find the following system of equations describing geodesic flows on the potential energy hypersurface:

$$\begin{aligned} \ddot{x}^1 &= a_1 \left\{ (\dot{x}^1)^2 - \sum_{\mu \neq 1, \mu=2}^6 (\dot{x}^\mu)^2 \right\} + 2\dot{x}^1 \left\{ a_2 \dot{x}^2 + a_3 \dot{x}^3 \right\}, \\ \ddot{x}^2 &= a_2 \left\{ (\dot{x}^2)^2 - \sum_{\mu=1, \mu \neq 2}^6 (\dot{x}^\mu)^2 \right\} + 2\dot{x}^2 \left\{ a_3 \dot{x}^3 + a_1 \dot{x}^1 \right\}, \\ \ddot{x}^3 &= a_3 \left\{ (\dot{x}^3)^2 - \sum_{\mu=1, \mu \neq 3}^6 (\dot{x}^\mu)^2 \right\} + 2\dot{x}^3 \left\{ a_1 \dot{x}^1 + a_2 \dot{x}^2 \right\}, \\ \ddot{x}^4 &= \dot{x}^4 \left\{ a_1 \dot{x}^1 + a_2 \dot{x}^2 + a_3 \dot{x}^3 \right\}, \\ \ddot{x}^5 &= \dot{x}^5 \left\{ a_1 \dot{x}^1 + a_2 \dot{x}^2 + a_3 \dot{x}^3 \right\}, \\ \ddot{x}^6 &= \dot{x}^6 \left\{ a_1 \dot{x}^1 + a_2 \dot{x}^2 + a_3 \dot{x}^3 \right\}, \end{aligned} \quad (11)$$

where $g(\{\bar{x}\}) = g_{11}(\{\bar{x}\}) = \dots = g_{66}(\{\bar{x}\})$ since the metric is the conformally Euclidean, in addition; $a_i(\{\bar{x}\}) = -(1/2)\partial_{x^i} \ln g(\{\bar{x}\})$.

In the system (11), the last three equations are integrated exactly:

$$\dot{x}^\mu = J_{\mu-3}/g(\{\bar{x}\}), \quad J_{\mu-3} = \text{const}_{\mu-3}, \quad (12)$$

where $\mu = \overline{4, 6}$.

Note that J_1, J_2 and J_3 are integrals of motion. They can be interpreted as projections of the total angular momentum of three-body system $J = \sqrt{J_1^2 + J_2^2 + J_3^2} = \text{const}$ on corresponding axis.

Substituting (12) into equations (11), we obtain the following system of non-linear second-order ordinary differential equations:

$$\begin{aligned}\ddot{x}^1 &= a_1 \{(\dot{x}^1)^2 - (\dot{x}^2)^2 - (\dot{x}^3)^2 - \Lambda^2\} + 2\dot{x}^1 \{a_2 \dot{x}^2 + a_3 \dot{x}^3\}, \\ \ddot{x}^2 &= a_2 \{(\dot{x}^2)^2 - (\dot{x}^3)^2 - (\dot{x}^1)^2 - \Lambda^2\} + 2\dot{x}^2 \{a_3 \dot{x}^3 + a_1 \dot{x}^1\}, \\ \ddot{x}^3 &= a_3 \{(\dot{x}^3)^2 - (\dot{x}^1)^2 - (\dot{x}^2)^2 - \Lambda^2\} + 2\dot{x}^3 \{a_1 \dot{x}^1 + a_2 \dot{x}^2\},\end{aligned}\quad (13)$$

where $\Lambda(\{\bar{x}\}) = (J/g\{\bar{x}\})^2$.

Doing designations;

$$\xi^1 = \dot{x}^1, \quad \xi^2 = \dot{x}^2, \quad \xi^3 = \dot{x}^3, \quad (14)$$

from the system (13) it is possible to obtain the following system of non-linear first order ODEs type of Ricatti:

$$\begin{aligned}\dot{\xi}^1 &= a_1 \{(\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 - \Lambda^2\} + 2\xi^1 \{a_2 \xi^2 + a_3 \xi^3\}, \\ \dot{\xi}^2 &= a_2 \{(\xi^2)^2 - (\xi^3)^2 - (\xi^1)^2 - \Lambda^2\} + 2\xi^2 \{a_3 \xi^3 + a_1 \xi^1\}, \\ \dot{\xi}^3 &= a_3 \{(\xi^3)^2 - (\xi^1)^2 - (\xi^2)^2 - \Lambda^2\} + 2\xi^3 \{a_1 \xi^1 + a_2 \xi^2\}.\end{aligned}\quad (15)$$

Thus, the system of equations (13) or the six order ODEs system (14)-(15) describes the dynamics of an imaginary point with an effective mass μ_0 , which moves on a Riemannian manifold; $\mathcal{M} = [\{\bar{x}\} \equiv (x^1, x^2, x^3) \in \mathcal{M}_t; g_{ij} = (E - U(\{\bar{x}\}))U_0^{-1}\delta_{ij} > 0]$.

Finally with regard to (9) and (12) we can get the reduced Hamiltonian:

$$\mathcal{H}(\{\bar{x}\}; \{\dot{\bar{x}}\}) = g_{\mu\nu}(\{\bar{x}\})p^\mu p^\nu = \frac{\mu_0}{2}g(\{\bar{x}\})\left\{\sum_{i=1}^3(\dot{x}^i)^2 + [J/g(\bar{x})]^2\right\}. \quad (16)$$

Substituting (16) into Hamilton equations:

$$\dot{x}^\mu = \frac{\partial \mathcal{H}}{\partial p^\mu}, \quad \dot{p}^\mu = -\frac{\partial \mathcal{H}}{\partial x^\mu}, \quad (17)$$

and by making simple calculations we can get the system of geodesic equations (13).

IV. THE CONDITIONS OF TRANSFORMATIONS OF 6D EUCLIDEAN SPACE TO THE 6D CONFORMAL-EUCLIDEAN SPACE

At obtaining of equations system (13), we have used some physical considerations that from the mathematical point of view are insufficiently rigorous, to argue that the dynamical

system (13) is equivalent to Newtonian problem of three-body. For a strict proof of equivalence of approaches, we need to prove that there is one-to-one mapping between two sets of coordinates; $\overline{\rho^1, \rho^6} = \{\rho\}$ and $\overline{x^1, x^6} = \{x\}$.

Let us consider two spaces $\mathbf{E}^6 \cong \mathbb{R}^6$ and \mathbf{M} which satisfy to the condition of one-to-one mapping; $\mathbf{E}^6 : \Leftrightarrow \mathbf{M}$. We will suppose that the Euclidean space \mathbf{E}^6 is defined by the set of coordinates $\{\rho\}$ and the metric tensor $\gamma_{\mu\nu}(\{\rho\})$, while the frame; $\{x\}$ and the metric tensor $g_{\mu\nu}(\{x\})$ respectively. The linear infinitesimal element in both coordinate systems can be represented as:

$$(ds)^2 = \gamma^{\alpha\beta}(\{\rho\})d\rho_\alpha d\rho_\beta = g_{\mu\nu}(\{x\})dx^\mu dx^\nu, \quad \alpha, \beta, \mu, \nu = \overline{1, 6}, \quad (18)$$

where the metric tensor $g_{\mu\nu}(\{x\})$ is defined as follows:

$$g_{\mu\nu}(\{x\}) = \gamma^{\alpha\beta}(\{\rho\})\rho_{\alpha;\mu}\rho_{\beta;\nu}, \quad (19)$$

where $\rho_{\alpha;\mu} = \partial\rho_\alpha/\partial x^\mu$.

Since tensor $g_{\mu\nu}(\{x\})$ is still defined in a rather arbitrary way we can impose additional conditions on it. In particular we will require that the metric tensor $g_{\mu\nu}(x)$ describes the conformal-Euclidean space; $g_{\mu\nu}(x) = g(\{x\})\delta_{\mu\nu}$ (see (9)). The last means that the following algebraic equations must be satisfied:

$$\gamma^{\alpha\beta}(\{\rho\})\rho_{\alpha;\mu}\rho_{\beta;\nu} = g(\{x\})\delta_{\mu\nu}. \quad (20)$$

As one can sure, the system of algebraic equations (20) is underdetermined since it consists from 21 equations while the number of unknown variables is 36. It is obvious that these equations are compatible, then the system (20) has an infinite number of real and complex solutions. These solutions form two different manifolds of 15th order. For a classical problem real solutions are the important ones, therefore below we will investigate properties of a real manifold. In a similar way we can obtain the system of algebraic equations for inverse transformations:

$$\gamma_{\alpha\beta}(\{\rho\})g^{-1}(\{\bar{x}\}) = x^\mu_{;\alpha}x^\nu_{;\beta}\delta_{\mu\nu}, \quad (21)$$

where $x^\mu_{;\alpha} = \partial x^\mu/\partial \rho^\alpha$.

It is obvious that if there are direct transformations then there are inverse transformations too.

Let us make new designations:

$$x_\mu = \rho_{1;\mu}, \quad y_\mu = \rho_{2;\mu}, \quad z_\mu = \rho_{3;\mu}, \quad u_\mu = \rho_{4;\mu}, \quad v_\mu = \rho_{5;\mu}, \quad w_\mu = \rho_{6;\mu}. \quad (22)$$

Taking into account the fact that the tensor; $g_{\mu\nu}(\{\bar{x}\})$ still is an arbitrary one, we can require fulfillment of following conditions for its elements:

$$\begin{aligned} x_4 = x_5 = x_6 = 0, \quad y_4 = y_5 = y_6 = 0, \\ z_4 = z_5 = z_6 = 0, \quad u_1 = u_2 = u_3 = 0, \\ v_1 = v_2 = v_3 = 0, \quad w_1 = w_2 = w_3 = 0. \end{aligned} \quad (23)$$

Using (6), (22) and conditions (23) from the equation (20) we can obtain two independent systems of algebraic equations:

$$\begin{aligned} x_1^2 + y_1^2 + \gamma^{33}z_1^2 &= g(\{\bar{x}\}), \\ x_2^2 + y_2^2 + \gamma^{33}z_2^2 &= g(\{\bar{x}\}), \\ x_3^2 + y_3^2 + \gamma^{33}z_3^2 &= g(\{\bar{x}\}), \\ x_1x_2 + y_1y_2 + \gamma^{33}z_1z_2 &= 0, \\ x_1x_3 + y_1y_3 + \gamma^{33}z_1z_3 &= 0, \\ x_2x_3 + y_2y_3 + \gamma^{33}z_2z_3 &= 0, \end{aligned} \quad (24)$$

and correspondingly:

$$\begin{aligned} \gamma^{44}u_4^2 + \gamma^{55}v_4^2 + \gamma^{66}w_4^2 + 2(\gamma^{45}u_4v_4 + \gamma^{46}u_4w_4 + \gamma^{56}v_4w_4) &= g(\{\bar{x}\}), \\ \gamma^{44}u_5^2 + \gamma^{55}v_5^2 + \gamma^{66}w_5^2 + 2(\gamma^{45}u_5v_5 + \gamma^{46}u_5w_5 + \gamma^{56}v_5w_5) &= g(\{\bar{x}\}), \\ \gamma^{44}u_6^2 + \gamma^{55}v_6^2 + \gamma^{66}w_6^2 + 2(\gamma^{45}u_6v_6 + \gamma^{46}u_6w_6 + \gamma^{56}v_6w_6) &= g(\{\bar{x}\}), \\ a_4u_4 + a_5v_4 + a_6w_4 &= 0, \\ b_4u_5 + b_5v_5 + b_6w_5 &= 0, \\ c_4u_6 + c_5v_6 + c_6w_6 &= 0. \end{aligned} \quad (25)$$

In equations (25) the following designations are made:

$$a_i = \gamma^{i4}u_5 + \gamma^{i5}v_5 + \gamma^{i6}w_5, \quad b_j = \gamma^{j4}u_6 + \gamma^{j5}v_6 + \gamma^{j6}w_6, \quad c_k = \gamma^{k4}u_4 + \gamma^{k5}v_4 + \gamma^{k6}w_4,$$

where $i = \overline{4, 6}$.

Note that solutions of algebraic systems (24) and (25) form two different manifolds of 3rd order, that are in one-to-one mapping with the potential energy hypersurface of the three-body system. Recall that analogical systems of algebraic equations can be obtained for inverse transformations. It is easy to see that coordinates which are defined by formulas (2) and (7) in general case do not satisfy to conditions of transformations (24) and (25) and correspondingly the system of equations (13) generally speaking is not equivalent to the Newtonian three-body problem. However, we have proved that there exists a system of local coordinates (8) that satisfy to systems of algebraic equations (24)-(25) and correspondingly with consideration of this circumstances, we can affirm that the equations system (13) is equivalent to the three-body Newtonian problem. Recall that in this case the equations (13) conserve their previous form and only the dependence of the potential from coordinates is changed.

The coordinate transformations between two sets of internal coordinates $\{\bar{\rho}\}$ and $\{\bar{x}\}$ are easy to represent in differential form:

$$\begin{aligned}\rho_1 &= \rho_1^0 + d\rho_1, & d\rho_1 &= x_1 dx^1 + x_2 dx^2 + x_3 dx^3, \\ \rho_2 &= \rho_2^0 + d\rho_2, & d\rho_2 &= y_1 dx^1 + y_2 dx^2 + y_3 dx^3, \\ \rho_3 &= \rho_3^0 + d\rho_3, & d\rho_3 &= z_1 dx^1 + z_2 dx^2 + z_3 dx^3,\end{aligned}\tag{26}$$

where $\{\bar{\rho}^0\} = (\rho_1^0, \rho_2^0, \rho_3^0)$ denotes the initial point.

Recall that at every stage of the evolution of a dynamical system the number of selection of the local coordinates system, as expected is unlimited. This fact in particular is reflected in infinite number of sets of solutions $[(x_1, ..), (y_1, ..), (z_1, ..)]_i$ of the algebraic system (24).

V. CLASSICAL MOVEMENT UNDER THE INFLUENCE OF QUANTUM FLUCTUATIONS

Let us assume that the dynamical system at the movement undergoes to the influence random forces, in particular to quantum fluctuations. In a mathematical sense, it is equivalent to the fact that the metric tensor and the corresponding coefficients in equations (15) are random functions:

$$Q_f : a_i(\{\bar{x}\}) \mapsto \tilde{a}_i(\{\bar{x}(s)\}) = \bar{a}_i(\{\bar{x}(s)\}) + \eta_i(s),$$

and

$$Q_f : \Lambda^2(\{\bar{x}\}) \mapsto \tilde{\Lambda}^2(\{\bar{x}(s)\}) = \bar{\Lambda}^2(\{\bar{x}(s)\}) + \eta_0(s),$$

where $\bar{a}_i(\{\bar{x}(s)\})$ and $\bar{\Lambda}^2(\{\bar{x}(s)\})$ are regular functions, the function Q_f displays random influences, the set of functions $\{\eta_0(s), \dots, \eta_3(s)\}$ denote random generators which will be refined below. It is obvious that the random component of \tilde{a}_i by a value is much more than a random member in the $\tilde{\Lambda}$, since \tilde{a}_i is the first derivative of metric tensor.

With consideration of said the system of equations (15) can be expanded and presented in the form of stochastic equations of Langevin type:

$$\dot{\xi}^i = A^i(\{\bar{\xi}\}|\{\bar{x}(s)\}) + \sum_{j=1}^3 B^{ij}(\{\bar{\xi}\}|\{\bar{x}(s)\})\eta_j(s) + O(\eta^2), \quad i = \overline{1, 3}, \quad (27)$$

where $\{\bar{\xi}\} = (\xi^1, \xi^2, \xi^3)$, in addition:

$$A^1(\{\bar{\xi}\}|\{\bar{x}(s)\}) = \bar{a}_1\{(\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 - \bar{\Lambda}^2\} + 2\xi^1(\bar{a}_2\xi^2 + \bar{a}_3\xi^3),$$

$$A^2(\{\bar{\xi}\}|\{\bar{x}(s)\}) = \bar{a}_2\{(\xi^2)^2 - (\xi^1)^2 - (\xi^3)^2 - \bar{\Lambda}^2\} + 2\xi^2(\bar{a}_3\xi^3 + \bar{a}_1\xi^1),$$

$$A^3(\{\bar{\xi}\}|\{\bar{x}(s)\}) = \bar{a}_3\{(\xi^3)^2 - (\xi^2)^2 - (\xi^1)^2 - \bar{\Lambda}^2\} + 2\xi^3(\bar{a}_1\xi^1 + \bar{a}_2\xi^2),$$

and, respectively;

$$B^{11} = (\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 - \bar{\Lambda}^2, \quad B^{12} = 2\xi^1\xi^2, \quad B^{13} = 2\xi^1\xi^3,$$

$$B^{21} = 2\xi^2\xi^1, \quad B^{22} = (\xi^2)^2 - (\xi^1)^2 - (\xi^3)^2 - \bar{\Lambda}^2, \quad B^{23} = 2\xi^2\xi^3,$$

$$B^{31} = 2\xi^3\xi^1, \quad B^{32} = 2\xi^3\xi^2, \quad B^{33} = (\xi^3)^2 - (\xi^2)^2 - (\xi^1)^2 - \bar{\Lambda}^2.$$

Note, that $\{\bar{x}(s)\}$ is the external parameter and is defined by solution the classical equations' system (13). In other words the variable ξ_i depends from $\{\bar{x}(s)\}$ parametrically and, accordingly it is correctly written in the form $\xi^i(s|\{\bar{x}(s)\})$. Recall that the cause of the stochastic motion of the system can be also random external force. At study the molecular dynamics will be more natural and important if we consider the quantum fluctuations. In this case the stochastic equations of motion (27) can be written in the form:

$$\dot{\xi}^i = A^i(\{\bar{\xi}\}|\{\bar{x}(s)\}) + \eta_i(s), \quad i = \overline{1, 3}, \quad (28)$$

where, it is natural to put, $\eta(s) = \eta_1(s) = \eta_2(s) = \eta_3(s)$.

If to assume that the average value of stochastic force is zero $\langle \eta(s) \rangle = 0$, then averaging of the stochastic equations (28) on relatively small scales of interval "s" leads to

the system of the classical equations (15). Note that the same we cannot say in respect to stochastic equations (27), since their averaging, generally speaking does not lead to the classical equations (15).

The joint probability density for the independent variables can be formally represented as:

$$P(\{\bar{\xi}\}, s|\{\bar{x}(s)\}) = \prod_{i=1}^3 \langle \delta[\xi^i(s|\{\bar{x}(s)\}) - \xi^i] \rangle. \quad (29)$$

Differentiating the expression (29) by variable "s" with considering the system of stochastic equations (27) and assuming that the stochastic generators satisfy to correlation properties of the *white noise*:

$$\langle \eta_i(s) \rangle = 0, \quad \langle \eta_i(s) \eta_j(s') \rangle = 2\epsilon_{ij} \delta(s - s'), \quad (30)$$

where the constant ϵ_{ij} describes the power of random fluctuations, for the joint probability density the following equation may be obtained²³ (more detail see²⁴):

$$\frac{\partial P}{\partial s} = \sum_{i=1}^3 \frac{\partial}{\partial \xi^i} (A^i P) + \sum_{i,j,l,k=1}^3 \epsilon_{ij} \frac{\partial}{\partial \xi^l} \left[B^{il} \frac{\partial}{\partial \xi^k} (B^{kj} P) \right]. \quad (31)$$

In the case of the quantum fluctuations, the equation of joint probability density is simplified (see equation (28)), and it may be found from (31) if to put $B^{ij} \equiv 1$ and $\epsilon_{ij} = \delta_{ij} \epsilon$, where δ_{ij} denotes the Kroneker symbol. Recall, that the power of the quantum fluctuations is defined by the expression, $\epsilon = \hbar \sqrt{\langle \omega^2 \rangle} / 2$, where $\langle \omega^2 \rangle$ is the frequency dispersion of randomly fluctuating virtual fields.

Thus, the equation (31) with the system of equations (13) and coordinate transformations (26) describes the classical multichannel scattering in the three-body system taking into account the quantum fluctuations. Finally, note that the function $P(\{\bar{\xi}\}, s|\{\bar{x}(s)\})$ is a probability of finding the momentum of the point mass with mass μ_0 in the range $[\xi, \xi + d\xi]$ and therefore it can be interpreted as $P = |\psi(\xi, s)|^2$, where $\psi(\xi, s)$ denotes the full wave function of the three-body system in the momentum representation, while $\xi \equiv \{\bar{\xi}\}$ is the 3D momentum in the units μ_0 .

VI. CONCLUSION

As it is well-known, the molecular dynamics in three-body system is often studied both in the framework of classical mechanics as well as by methods of the quantum mechanics.

This is due to the fact that molecular systems depending on values initial parameters (collision energy, mass of particles etc.) can demonstrate all diversity of classical and quantum motions. Despite the fact that the quantum mechanics is a more general representation than the classical mechanics, nonetheless as claimed by many researchers, often the classical calculations more accurately describe the nature of molecular dynamics than the quantum calculations. This fact is not accidental, since the molecular systems as a rule in large regions of phase space often demonstrate chaotic motion, which is not anti-aliased by the quantum uncertainties, by given above reasons.

For solving the above stated problems, we made extension of the classical three-body problem, assuming that the metric of the conform-Euclidean space has a random component. In result of this we obtained the system of Langevin type SDEs (27) and also (28), which describe dynamics the three-body system under the influence of random forces, origin of which can be different, in particular can be quantum fluctuations. Assuming, that the fluctuations satisfy the correlation conditions of the white noise and, using SDEs (27) we get the evolution equation for the quantum probability currents in the form of a partial second order differential equation (31). Note that the equation (31) can be solved combining with the classical equations of motion (15) and algebraic equations (24) taking into account coordinate transformations (26).

Now obviously, if the solution of the classical system has a chaotic behavior, then this will be reflected on properties of quantum probability currents and correspondingly on the wavefunction of bodies system. To assess the nature of the propagation of quantum probability currents arising at the process of multichannel scattering in the three-body system, it is useful to define the criterion of the discrepancy of two arbitrary probability flows. Following to definition of the Kullback-Leibler relatively the distance between two continuous distributions²⁵, $P_a = P(\{\bar{\xi}\}, s|\{\bar{x}_a(s)\})$ and $P_b = P(\{\bar{\xi}\}, s|\{\bar{x}_b(s)\})$, we can determine the distance between the two tubes probabilistic quantum currents in the following way:

$$D_{ab}(s) = \int_{\mathcal{M}_t} P(\{\bar{\xi}\}, s|\{\bar{x}_a(s)\}) \ln \frac{P(\{\bar{\xi}\}, s|\{\bar{x}_a(s)\})}{P(\{\bar{\xi}\}, s|\{\bar{x}_b(s)\})} d\xi^1 d\xi^2 d\xi^3,$$

where it is assumed that $\{\bar{x}_a(s) \in \mathcal{M}_t$ and $\{\bar{x}_b(s)\} \in \mathcal{M}_t$ are two different trajectories. If at some point in time s_0 , these trajectories come out from the single point $\{\bar{x}^0\}$, then we say that in this moment of time the probability distributions, and respectively the wave functions have zero distance or they are the same. The trajectories as well as the corresponding

distributions can diverge during the time. In case the distance between two flows depending on time exponentially grow, i.e., $D_{ab}(s) \sim e^{ks}$, where $k > 0$ is some constant, then there is every reason to believe that the quantum system wavefunction is chaotic and hence the system is quantum-chaotic.

Lastly important to note that the full quantum theory of multichannel scattering in the three-body system allowing the emergence of quantum chaos in the wave function, can be constructed based on Schrödinger equation with the Hamiltonian (16), also taking into account the classical equations (15), (24) and (26). Recall that the system of the classical equations in this case is responsible for the topological peculiarities of tubes of the quantum probabilistic currents and transitions between asymptotic channels.

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